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ON SECTIONAL GENUS OF QUASI-POLARIZED 3-FOLDS

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Abstract. Let X be a smooth projective variety over $\mathbb{C}$ and L a nef-big (resp. ample) divisor on X. Then (X, L) is called a quasi-polarized (resp. polarized) manifold. Then we conjecture that $g(L) \geq q(X)$, where $g(L)$ is the sectional genus of L and $q(X) = \dim H^1(\mathcal{O}_X)$ is the irregularity of X. In general it is unknown whether this conjecture is true or not, even in the case of $\dim X = 2$. For example, this conjecture is true if $\dim X = 2$ and $\dim H^0(L) > 0$. But it is unknown if $\dim X \geq 3$ and $\dim H^0(L) > 0$. In this paper, we prove $g(L) \geq q(X)$ if $\dim X = 3$ and $\dim H^0(L) \geq 2$. Furthermore we classify polarized manifolds (X, L) with $\dim X = 3$, $\dim H^0(L) \geq 3$, and $g(L) = q(X)$.

Introduction

Let (X, L) be a quasi-polarized manifold with $\dim X = n$. For this (X, L), the sectional genus $g(L)$ is defined by the following formula:

$$g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1}.$$

Then there is the following conjecture which is interesting and difficult.

Conjecture. *Let (X, L) be a quasi-polarized manifold. Then $g(L) \geq q(X)$, where $q(X) = \dim H^1(\mathcal{O}_X)$.*

For this Conjecture, there are some results (see [Fk1], [Fk2], [Fk3]). But it is unknown whether this Conjecture is true or not, even in the case of $\dim X = 2$. If $\dim X = 2$, then this Conjecture is true if $h^0(L) = \dim H^0(L) > 0$. This proof is easy (see [Fk1]). But if $\dim X \geq 3$, it is unknown whether this Conjecture is true or not, even in the case $h^0(L) > 0$. In this paper, we study the case in which $\dim X = 3$ and $h^0(L) \geq 2$.

In the paper [Fk4], we proved that $g(L) \geq (2/3)q(X) + 1/3$ if $\kappa(X) \geq 0$ and $h^0(L) \geq 2$. But we can improve this result.

First we will prove the following theorem.

Theorem 2.1. *Let (X, L) be a quasi-polarized 3-fold with $h^0(L) \geq 2$. Then $g(L) \geq q(X)$.*

The method of the proof of Theorem 2.1 is thought to be the best way to prove the Conjecture if $h^0(L) \geq 2$, and we can find out that this Conjecture is related to the minimal model problem (in particular the Flip Conjecture). (See Theorem 2.5.)

Furthermore if $\dim X = 3$, $h^0(L) \geq 3$, and $g(L) = q(X)$, then we can classify the type of polarized 3-folds (X, L) as follows.

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Theorem 2.12. *Let (X, L) be a polarized 3-fold with $h^0(L) \geq 3$. If $g(L) = q(X)$, then (X, L) is one of the following types:*

1. $\Delta(L) = 0$*;*
2. (X, L) *is a scroll over a curve.*

The case in which $h^0(L) = 1$ will be studied in a forthcoming paper.

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1. Preliminaries

Definition 1.1. The pair (X, L) is called a quasi-polarized (resp. polarized) manifold if X is a smooth projective variety over $\mathbb{C}$ and L is a nef-big (resp. an ample) line bundle. (f, X, Y) is called a fiber space if X and Y are smooth projective varieties over $\mathbb{C}$ with $\dim X > \dim Y \geq 1$ and f is a surjective morphism $X \to Y$ with connected fibers. (f, X, Y, L) is called a quasi-polarized (resp. polarized) fiber space if (f, X, Y) is a fiber space and L is a nef-big (resp. an ample) line bundle.

Definition 1.2. (1) Let (X_1, L_1) and (X_2, L_2) be quasi-polarized manifolds, where X_i may have singularities for i =1, 2. Then (X_1, L_1) and (X_2, L_2) are said to be birationally equivalent if there is another variety G with birational morphisms $g_i : G \to X_i$ $(i = 1, 2)$ such that $g_1^* L_1 = g_2^* L_2$.

(2) Let (f_1, X_1, Y, L_1) and (f_2, X_2, Y, L_2) be quasi-polarized fiber spaces, where X_i may have singularities for i =1, 2. Then (f_1, X_1, Y, L_1) and (f_2, X_2, Y, L_2) are said to be birationally equivalent if there is another variety G with birational morphisms $g_i : G \to X_i$ $(i = 1, 2)$ such that $g_1^* L_1 = g_2^* L_2$ and $f_1 \circ g_1 = f_2 \circ g_2$.

Theorem 1.3. *Let (f, X, C, L) be a polarized fiber space with $\dim X = n \geq 3$. Then*

(1) $K_{X/C} + (n-1)L$ is nef unless (f, X, C, L) is a scroll.

(2) $g(L) \geq g(C)$ holds and if this equality holds, then (f, X, C, L) is a scroll.

Proof. See Theorem 1.1.2, Theorem 1.2.1, and Theorem 1.4.2 in [Fk2]. □

Theorem 1.4. *Let (X, L) be a quasi-polarized 3-fold. Then there exists a quasi-polarized variety (X', L') which is birationally equivalent to (X, L) and satisfies one of the following conditions:*

1. $K_{X'} + 2L'$ *is nef for the canonical* $\mathbb{Q}$*-bundle* $K_{X'}$*,*
2. $\Delta(L') = 0$*,*
3. (X', L') *is a scroll over a curve,*

where X' is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities.

Proof. See Theorem 4.2 in [Fj2]. □

Theorem 1.5. *Let (f, X, C, L) be a quasi-polarized fiber space with $\dim X = 3$ and $\dim C = 1$. Then there exists a quasi-polarized fiber space (f', X', C, L') which is birationally equivalent to (f, X, C, L) such that (f', X', C, L') satisfies one of the following conditions:*

1. $K_{X'} + 2L'$ *is* f'*-nef,*
2. (f', X', C, L') *is a scroll,*

where X' is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities.

Proof. See Theorem 1.3 in [Fk3]. □

Theorem 1.6. *Let* (f, X, C, L) *be a quasi-polarized fiber space with* $\dim X = 3$ *and* $\dim C = 1$. *Then* $g(L) \geq g(C)$.

Proof. See Theorem 1.4 in [Fk3]. □

Remark 1.7. If the Flip Conjecture (see [KMM]) is true for $n = \dim X$, then Theorems 1.4, 1.5, and 1.6 are true for $n = \dim X$.

Lemma 1.8. *Let* (f, X, Y, L) *be a quasi-polarized fiber space, where* X *is a normal projective variety of* $\dim X \geq 2$ *with only* $\mathbb{Q}$*-factorial canonical singularities. Assume that* $K_{X/Y} + tL$ *is* f*-nef , where* t *is a positive integer. Then* $(K_{X/Y} + tL)L^{n-1} \geq 0$.

Moreover if $\dim Y = 1$, *then* $K_{X/Y} + tL$ *is nef.*

Proof. See Lemma 0.2 in [Fk3]. □

Definition 1.9. Let X and Y be n-dimensional projective manifolds, L an ample divisor on X, and $\pi : X \to Y$ one point blowing up. Let E be a π-exceptional reduced divisor. Then π is said to be simple blowing up if $(E, L_E) \cong (\mathbb{P}^{n-1}, \mathcal{O}(1))$.

Theorem 1.10. *Let* (X, L) *be a polarized manifold with* $\dim X = n$. *If* $K_X + (n-1)L$ *is not nef, then* (X, L) *is one of the following types:*

(1) $\Delta(L) = 0$, *where* $\Delta(L) = n + L^n - h^0(L)$. *(See* [Fj0]*.)*

(2) (X, L) *is a scroll over a curve.*

Proof. See [Fj1] or [I]. □

Remark 1.11. Let (X, L) be a polarized manifold with $\dim X = n$. If $K_X + (n-1)L$ is not nef, then $g(L) \geq q(X)$.

Theorem 1.12. *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 3$. *Assume that* $K_X + (n-1)L$ *is nef. If* $K_X + (n-2)L$ *is not nef, then* (X, L) *is one of the following types:*

(a) (X, L) *is obtained by some simple blowing up of another polarized manifold.*

(b0) (X, L) *is a Del Pezzo manifold with* $b_2(X) = 1$, *or* $(\mathbb{P}^3, \mathcal{O}(j))$ *with* $j = 2$ *or* 3, $(\mathbb{P}^4, \mathcal{O}(2))$, *or a hyperquadric in* $\mathbb{P}^4$ *with* $L = \mathcal{O}(2)$.

(b1) There is a fibration $\Phi : X \to W$ *over a curve* W *with one of the following properties:*

(b1-v) $(F, L_F) \cong (\mathbb{P}^2, \mathcal{O}(2))$ *for any fiber* F *of* Φ.

(b1-q) Every fiber F *of* Φ *is an irreducible hyperquadric in* $\mathbb{P}^n$ *having only isolated singularities.*

(b2) (X, L) *is a scroll over a smooth surface.*

Proof. See [Fj1] or [I]. □

Theorem 1.13. *Let* (X, L) *be a quasi-polarized manifold. If* L *is ample or* $\dim X \leq 3$, *then* $g(L) \geq 0$.

Proof. See [Fj1], [Fj2], or [I]. □

Remark 1.14. If the Flip Conjecture (see [KMM]) is true for $n = \dim X$, then Theorem 1.13 is true for $n = \dim X$.

Lemma 1.15. *Let* (X, L) *be a quasi-polarized manifold. If* $\Delta(L) = 0$, *then* $q(X) = 0$.

Proof. By Theorem (1.1) in [Fj2], there exist a variety W, a birational morphism $f : X \to W$, and a very ample line bundle H on W such that $L = f^*H$ and $\Delta(H) = 0$. Then W is normal and has only rational singularities (see Corollary (5.17) in [Fj0]). Since X is smooth, we have $q(X) = h^1(\mathcal{O}_W)$. But $h^1(\mathcal{O}_W) = 0$ in this case (see Chapter I in [Fj0]). Hence $q(X) = 0$. □

Theorem 1.16. *Let (X, L) be a quasi-polarized manifold. If L is ample or* $\dim X \leq 3$*, then $\Delta(L) = 0$ if and only if $g(L) = 0$.*

Proof. See [Fj1] and [Fj2]. □

Definition 1.17. Let (X, L) be a quasi-polarized surface. We say that (X, L) is L-minimal if $L.E > 0$ for any (-1)-curve E on X. For any quasi-polarized surface (X, L) , there exists a birational morphism $\rho : (X, L) \to (X_0, L_0)$ such that $L = \rho^* L_0$ and (X_0, L_0) is L_0-minimal. Then we call (X_0, L_0) an L-minimalization of (X, L).

Theorem 1.18 (classical). *Let (X, L) be a polarized manifold with* $\dim X = n$*. Assume that $|L|$ has no base point. If $g(L) = q(X)$, then (X, L) is one of the following types.*

(1) $\Delta(L) = 0$*.*

(2) (X, L) is a scroll over a curve.

Proof. See [S] or [S-V]. □

Theorem 1.19. *Let (X, L) be a polarized manifold with* $\dim X = n$*. Assume that $|L|$ has no base point. Then $g(L) \geq 2q(X) - 1$ unless (X, L) is a scroll over a curve C with $g(C) \geq 2$.*

Proof. We take $(n - 2)$ general elements of $|L|$. By cutting these elements, there exists a polarized surface (S, L_S) such that $g(L) = g(L_S)$ and $q(X) = q(S)$. So we may consider (S, L_S).

(1) The case in which $\kappa(S) \geq 0$.

Then $g(L_S) \geq 2q(S) - 1$ by Corollary 3.2 in [Fk4]. Hence $g(L) = g(L_S) \geq 2q(S) - 1 = 2q(X) - 1$.

(2) The case in which $\kappa(S) = -\infty$.

If $q(S) = 0$, then $g(L) = g(L_S) \geq 0 = 2q(S) = 2q(X)$. So we may assume $q(S) \geq 1$. Then $K_S^2 \leq 8(1 - q(S))$. On the other hand,

$$\begin{aligned}(K_S + L)^2 &= K_S^2 + 2(K_S + L_S)L_S - L_S^2 \\ &\leq 8(1 - q(S)) + 4(g(L_S) - 1) - L_S^2 \\ &= 4(g(L_S) - 2q(S) + 1) - L_S^2.\end{aligned}$$

If $K_S + L_S$ is nef, then $(K_S + L_S)^2 \geq 0$. So we have $g(L_S) \geq 2q(S)$.

If $K_S + L_S$ is not nef, then (S, L_S) is a scroll over a curve by Mori theory (see [Fk1]). Hence $g(L) = g(L_S) \geq 2q(S) = 2q(X)$ unless (S, L_S) is a scroll over a curve. On the other hand, if (S, L_S) is a scroll over a curve with $q(S) \geq 1$, then (X, L) is a scroll over a curve C by Bădescu ([B1], [B2], [B3]). (See also (5.5) in [B-S].)

If $g(C) \leq 1$, then $g(L) \geq 2q(X) - 1$. This completes the proof of Theorem 1.19. □

Notation 1.20. Let D be an effective divisor on a smooth projective variety X and $D = \sum_i a_i D_i$ its prime decomposition, where $a_i \geq 1$ for any i.

Then we write $D_{\text{red}} = \sum_i D_i$.

2. Main Results

In this section we consider (quasi-)polarized 3-folds with $h^0(L) \geq 2$.

Theorem 2.1. *Let (X, L) be a quasi-polarized 3-fold with $h^0(L) \geq 2$. Then $g(L) \geq q(X)$.*

Proof. By Theorem 1.4, there exists a quasi-polarized variety (X', L') which is birationally equivalent to (X, L) and satisfies one of the following conditions:

1. $K_{X'} + 2L'$ is nef for the canonical $\mathbb{Q}$-bundle $K_{X'}$,
2. $\Delta(L') = 0$,
3. (X', L') is a scroll over a curve,

where X' is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities.

Since $g(L) = g(L')$ and $q(X) = q(X')$, we may assume that X has only $\mathbb{Q}$-factorial terminal singularities and satisfies one of the above conditions.

If (X, L) is the type (2), then $q(X) = 0$ by Lemma 1.15. Since $g(L) \geq 0$ for any quasi-polarized 3-fold by Theorem 1.13, we obtain that $g(L) \geq q(X)$ in this case.

If (X, L) is the type (3), then $g(L) = q(X)$ by easy calculation.

So we may assume that $K_X + 2L$ is nef. Let $\pi : \widetilde{X} \to X$ be a resolution of X such that $\widetilde{X} \setminus \pi^{-1}(\mathrm{Sing}(X)) \cong X \setminus \mathrm{Sing}(X)$, and $\widetilde{L} = \pi^*(L)$. Then $h^0(L) = h^0(\widetilde{L}) \geq 2$. Let Λ be a linear pencil which is contained in $|\widetilde{L}|$ such that $\Lambda = \Lambda_M + Z$, where Λ_M is the movable part of Λ and Z is the fixed part of $|\widetilde{L}|$. We will make a fiber space by using this Λ. Let $\varphi : \widetilde{X} \dashrightarrow \mathbb{P}^1$ be the rational map associated with Λ_M, and $\theta : \widetilde{X}' \to \widetilde{X}$ an elimination of indeterminacy of φ. So we obtain a surjective morphism $\varphi' : \widetilde{X}' \to \mathbb{P}^1$. If necessary, we take the Stein factorization $\delta : C \to \mathbb{P}^1$ of φ'. Then we have a fiber space $f' : \widetilde{X}' \to C$ such that $\varphi' = \delta \circ f'$. We consider this quasi-polarized fiber space $(f', \widetilde{X}', C, \theta^* \widetilde{L})$.

Case (1). $g(C) \geq 1$. In this case, $\theta = \mathrm{id}$. So $\widetilde{X}' = \widetilde{X}$. Then

$$g(L) = g(\widetilde{L}) = g(C) + \frac{1}{2}(K_{\widetilde{X}/C} + 2\widetilde{L})(\widetilde{L})^2 + (\widetilde{L}^2 F' - 1)(g(C) - 1),$$

where F' is a general fiber of f'.

By Theorem 1.5, there exists a quasi-polarized fiber space (f_1, X_1, C, L_1) which is birationally equivalent to $(f', \widetilde{X}, C, \widetilde{L})$ such that (f_1, X_1, C, L_1) satisfies one of the following conditions:

1. $K_{X_1} + 2L_1$ is f_1-nef,
2. (f_1, X_1, C, L_1) is a scroll,

where X_1 has a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities.

If (f_1, X_1, C, L_1) is the type (2), then $g(L) = g(L_1) = g(C) = q(X_1) = q(X)$. So we may assume that $K_{X_1} + 2L_1$ is f_1-nef. Then by Lemma 1.8, $(K_{X_1/C} + 2L_1)$ is nef. We take a general member B of $|\widetilde{L}|$. Then $B \equiv aF' + Z$, where F' is a general fiber of f' and $a = \deg \delta$.

Claim 2.2. $(K_{\widetilde{X}/C}+2\widetilde{L})\widetilde{L}(\widetilde{L}-tF')\geq 0$ *for any natural number* $t\leq a$.

Proof. By Theorem 1.5, there exist a smooth projective variety G and birational morphisms $\varepsilon_1 : G\to\widetilde{X}$ and $\varepsilon_2 : G\to X_1$ such that $\varepsilon_1^*(\widetilde{L})=\varepsilon_2^*(L_1)$.

Then

$$\begin{aligned}&(K_{\widetilde{X}/C}+2\widetilde{L})\widetilde{L}(\widetilde{L}-tF')\\&=(K_{G/C}+2\varepsilon_1^*(\widetilde{L}))(\varepsilon_1^*(\widetilde{L}))(\varepsilon_1^*(\widetilde{L}-tF'))\\&=(\varepsilon_2^*(K_{X_1/C}+2L_1)+E_{\varepsilon_2})(\varepsilon_2^*L_1)(\varepsilon_2^*(L_1-tF_1))\\&=\varepsilon_2^*(K_{X_1/C}+2L_1)(\varepsilon_2^*L_1)(\varepsilon_2^*(L_1-tF_1)),\end{aligned}$$

where E_{ε_2} is an ε_2-exceptional $\mathbb{Q}$-divisor and F_1 is a general fiber of f_1.

We remark that $\varepsilon_1^*(\widetilde{L}-tF')=\varepsilon_2^*(L_1-tF_1)$, $\varepsilon_1^*(\widetilde{L}-tF')$ is numerically equivalent to an effective divisor, and $K_{X_1/C}+2L_1$ is nef. Hence we obtain

$$\varepsilon_2^*(K_{X_1/C}+2L_1)(\varepsilon_2^*L_1)(\varepsilon_2^*(L_1-tF_1))\geq 0.$$

This completes the proof of this Claim. □

Therefore

$$g(\widetilde{L})\geq g(C)+(K_{\widetilde{X}/C}+2\widetilde{L})\widetilde{L}F'$$

since $\widetilde{L}^2F'\geq 1$, $g(C)\geq 1$, and $a\geq 2$. So we obtain

$$\begin{aligned}g(\widetilde{L})&\geq g(C)+(K_{F'}+2\widetilde{L}_{F'})\widetilde{L}_{F'}\\&=g(C)+(K_{F'}+\widetilde{L}_{F'})\widetilde{L}_{F'}+\widetilde{L}_{F'}^2\\&=g(C)+2g(\widetilde{L}_{F'})+\widetilde{L}_{F'}^2-2\\&=g(C)+g(\widetilde{L}_{F'})+g(\widetilde{L}_{F'})+\widetilde{L}_{F'}^2-2.\end{aligned}$$

Since $h^0(\widetilde{L}_{F'})\geq 1$ and $\dim F'=2$, we have $g(\widetilde{L}_{F'})\geq q(F')$. Hence

$$\begin{aligned}g(\widetilde{L})&\geq g(C)+q(F')+g(\widetilde{L}_{F'})+\widetilde{L}_{F'}^2-2\\&\geq q(\widetilde{X})+g(\widetilde{L}_{F'})+\widetilde{L}_{F'}^2-2.\end{aligned}$$

If $g(\widetilde{L}_{F'})=0$, then $q(F')=0$ and $q(\widetilde{X})=g(C)$. On the other hand, $g(\widetilde{L})\geq g(C)$ by Theorem 1.6. So $g(L)=g(\widetilde{L})\geq g(C)=q(\widetilde{X})=q(X)$.

If $g(\widetilde{L}_{F'})\geq 1$, then $g(\widetilde{L})\geq q(\widetilde{X})$ since $\widetilde{L}_{F'}^2\geq 1$.

Case (2). $g(C)=0$. Let $\widetilde{D}=\theta(F')$ and $D=\pi(\widetilde{D})$. We remark that $D\neq 0$ because F' is a general fiber. Then $\widetilde{L}-\widetilde{D}$ is linearly equivalent to an effective divisor. We put $\gamma=\pi\circ\theta$. Then

$$\begin{aligned}g(L)=g(\widetilde{L})=g(\theta^*\widetilde{L})&=1+\frac{1}{2}(K_{\widetilde{X}'}+2\theta^*\widetilde{L})(\theta^*\widetilde{L})^2\\&=1+\frac{1}{2}(\theta^*(K_{\widetilde{X}}+2\widetilde{L}))(\theta^*\widetilde{L})^2\\&=1+\frac{1}{2}(\theta^*(K_{\widetilde{X}})+2\gamma^*L)(\theta^*\widetilde{L})^2\\&=1+\frac{1}{2}\gamma^*(K_X+2L)(\theta^*\widetilde{L})^2\\&\geq 1+\frac{1}{2}\gamma^*(K_X+2L)(\theta^*\widetilde{L})F'\end{aligned}$$

because K_X+2L is nef.

Let $K_{\widetilde{X}} = \pi^*(K_X) + \sum a_i \widetilde{E}_i$, where $\widetilde{E}_i$ is a π-exceptional effective divisor.

Claim 2.3. $(\theta^* \widetilde{E}_i)(\theta^* \widetilde{L})(F') = 0$.

Proof.

$$\begin{aligned}(\theta^* \widetilde{E}_i)(\theta^* \widetilde{L})(F') &= (\widetilde{E}_i)(\widetilde{L})\widetilde{D} \\ &= (\widetilde{E}_i|_{\widetilde{D}})(\pi|_{\widetilde{D}})^*(L|_D).\end{aligned}$$

By the definition of π, $\pi(\widetilde{E}_i) \subset \mathrm{Sing}(X)$. On the other hand, $\operatorname{Codim}\mathrm{Sing}(X) \geq 3$ because X has only terminal singularities. Hence $\dim(\pi|_{\widetilde{D}})(\widetilde{E}_i)$ is at most 0 in D. We remark that $\dim D = 2$. Therefore $(\widetilde{E}_i|_{\widetilde{D}})(\pi|_{\widetilde{D}})^*(L|_D) = 0$. This completes the proof of Claim 2.3. □

Hence we obtain

$$\begin{aligned}g(L) &\geq 1 + \frac{1}{2}\theta^*(\pi^* K_X + 2\widetilde{L})(\theta^* \widetilde{L})F' \\ &= 1 + \frac{1}{2}\theta^*(\pi^* K_X + \sum a_i \widetilde{E}_i + 2\widetilde{L})(\theta^* \widetilde{L})F' \\ &= 1 + \frac{1}{2}\theta^*(K_{\widetilde{X}} + 2\widetilde{L})(\theta^* \widetilde{L})F'.\end{aligned}$$

Since F' is nef and $\widetilde{L} - \widetilde{D}$ is linearly equivalent to an effective divisor, we obtain

$$\begin{aligned}g(L) &\geq 1 + \frac{1}{2}\theta^*(K_{\widetilde{X}} + 2\widetilde{L})(\theta^* \widetilde{L})F' \\ &= 1 + \frac{1}{2}(\theta^*(K_{\widetilde{X}}) + \theta^*(\widetilde{D}) + \theta^*(\widetilde{L} - \widetilde{D}) + \theta^* \widetilde{L})(\theta^* \widetilde{L})F' \\ &\geq 1 + \frac{1}{2}(\theta^*(K_{\widetilde{X}} + \widetilde{D}) + \theta^* \widetilde{L})(\theta^* \widetilde{L})F'.\end{aligned}$$

Let $\theta_i : X_i \to X_{i-1}$ be a blowing up at a smooth center B_{i-1} and $\theta = \theta_1 \circ \cdots \circ \theta_t$. Let $X_0 = \widetilde{X}$ and $X_t = \widetilde{X}'$. Let D_i be the strict transform of D_{i-1}, $D_0 = \widetilde{D}$, and $D_t = F'$. Let $\theta_i^* D_{i-1} = D_i + d_i E_i$, where E_i is a θ_i-exceptional effective reduced divisor. Then $K_{X_i} = \theta_i^*(K_{X_{i-1}}) + a_i E_i$, where $a_i = 1$ if $\dim B_{i-1} = 1$ and $a_i = 2$ if $\dim B_{i-1} = 0$.

Claim 2.4.

$$(\theta_t^* \circ \cdots \circ \theta_k^*)(K_{X_{k-1}} + D_{k-1})(\theta^* \widetilde{L})F' \geq (\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(K_{X_k} + D_k)(\theta^* \widetilde{L})F'.$$

Proof. (1) The case in which $\dim B_{k-1} = 0$.

In this case

$$\begin{aligned}&(\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(E_k)(\theta^* \widetilde{L})F' \\ &= (E_k)((\theta_k^* \circ \cdots \circ \theta_1^*)(\widetilde{L}))D_k \\ &= (E_k|_{D_k})(\theta_k|_{D_k})^*((\theta_{k-1}^* \circ \cdots \circ \theta_1^* \widetilde{L})|_{D_{k-1}}) \\ &= 0.\end{aligned}$$

Hence

$$
\begin{aligned}
&(\theta_t^* \circ \cdots \circ \theta_k^*)(K_{X_{k-1}} + D_{k-1})(\theta^*\widetilde{L})F' \\
&= (\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(\theta_k^*(K_{X_{k-1}}) + \theta_k^* D_{k-1})(\theta^*\widetilde{L})F' \\
&= (\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(\theta_k^*(K_{X_{k-1}}) + 2E_k + D_k)(\theta^*\widetilde{L})F' \\
&= (\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(K_{X_k} + D_k)(\theta^*\widetilde{L})F'.
\end{aligned}
$$

(2) The case in which $\dim B_{k-1} = 1$.

In this case, $(\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(E_k)(\theta^*\widetilde{L})F' \geq 0$ since F' is nef. In particular, $(\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(E_k)(\theta^*\widetilde{L})F' = 0$ if $d_k = 0$ (that is, $D_k = \theta_k^*(D_{k-1})$) because

$$
\begin{aligned}
&(\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(E_k)(\theta^*\widetilde{L})F' \\
&= (E_k)((\theta_k^* \circ \cdots \circ \theta_1^*)(\widetilde{L}))D_k \\
&= (E_k)((\theta_k^* \circ \cdots \circ \theta_1^*)(\widetilde{L}))\theta_k^* D_{k-1} \\
&= 0.
\end{aligned}
$$

Hence by the above argument

$$
\begin{aligned}
&(\theta_t^* \circ \cdots \circ \theta_k^*)(K_{X_{k-1}} + D_{k-1})(\theta^*\widetilde{L})F' \\
&= (\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(\theta_k^*(K_{X_{k-1}}) + D_k + d_k E_k)(\theta^*\widetilde{L})F' \\
&\geq (\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(\theta_k^*(K_{X_{k-1}}) + E_k + D_k)(\theta^*\widetilde{L})F' \\
&\geq (\theta_t^* \circ \cdots \circ \theta_{k+1}^*)(K_{X_k} + D_k)(\theta^*\widetilde{L})F'.
\end{aligned}
$$

This completes the proof of this Claim. □

By Claim 2.4,

$$
\begin{aligned}
g(L) &\geq 1 + \frac{1}{2}(\theta^*(K_{\widetilde{X}} + \widetilde{D}) + \theta^*\widetilde{L})(\theta^*\widetilde{L})F' \\
&\geq 1 + \frac{1}{2}(K_{\widetilde{X}'} + F' + \theta^*\widetilde{L})(\theta^*\widetilde{L})F' \\
&= g(\theta^*\widetilde{L}|_{F'}).
\end{aligned}
$$

Since $h^0(\theta^*\widetilde{L}|_{F'}) \geq 1$ and $\dim F' = 2$, we have $g(\theta^*\widetilde{L}|_{F'}) \geq q(F')$. On the other hand, $q(F') \geq q(\widetilde{X}') = q(X)$ in this case. Therefore $g(L) \geq g(\theta^*\widetilde{L}|_{F'}) \geq q(F') \geq q(X)$. This completes the proof of Theorem 2.1. □

By the same argument as the proof of Theorem 2.1, we can prove the following Theorem.

Theorem 2.5. *Let (X, L) be a quasi-polarized manifold with $\dim X = n \geq 4$ and $h^0(L) \geq 2$. Assume that the Flip Conjecture (see* [KMM]*) is true and (X, L) is not birationally equivalent to a quasi-polarized variety (X', L') such that $\Delta(L') = 0$ or (X', L') is a scroll over a curve, where X' is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities. Then there exists a quasi-polarized fiber space (f, Y, C, A) with $\dim Y = n$, $\dim C = 1$, and $h^0(A) \geq 2$ such that Y is birationally equivalent to X, $g(L) \geq g(A|_F) + g(C)$ and $q(F) + g(C) \geq q(Y) = q(X)$, where F is a general fiber of f.*

We fix the notation which is used in the following theorems.

Notation 2.6. Let (X, L) be a polarized manifold with $\dim X = n \geq 3$ and $h^0(L) \geq 2$. Let Λ be a linear pencil which is contained in $|L|$ such that $\Lambda = \Lambda_M + Z$, where Λ_M is the movable part of Λ and Z is the fixed part of $|L|$. We will make a fiber space by using this Λ. Let $\varphi : X \dashrightarrow \mathbb{P}^1$ be the rational map associated with Λ_M and $\theta : X' \to X$ be an elimination of indeterminacy of φ. So we obtain a surjective morphism $\varphi' : X' \to \mathbb{P}^1$. If necessary, we take the Stein factorization $\delta : C \to \mathbb{P}^1$ of φ'. Then we have a fiber space $f' : X' \to C$ such that $\varphi' = \delta \circ f'$. Let $a = \deg \delta$ and F' be a general fiber of f'.

We remark that we can prove the following Theorem for a polarized manifold (X, L) of $\dim X = n \geq 4$.

Theorem 2.7. *Let (X, L) be a polarized manifold with $\dim X = n \geq 4$ and $h^0(L) \geq 2$. We use Notation 2.6. Then*

$$g(L) \geq g(C) + g(\theta^* L_{F'})$$

unless $\Delta(L) = 0$ or (X, L) is a scroll over a curve.

Proof. Case (a): $g(C) \geq 1$.

Then we remark that $\theta = \mathrm{id}$. We consider the polarized fiber space (f', X, C, L). Then

$$g(L) = g(C) + \frac{1}{2}(K_{X/C} + (n-1)L)L^{n-1} + (L^{n-1}F' - 1)(g(C) - 1),$$

where F' is a general fiber of f'.

If $K_{X/C} + (n-1)L$ is not nef, then (f', X, C, L) is scroll by Theorem 1.3. Hence we may assume that $K_{X/C} + (n-1)L$ is nef. Since $a \geq 2$, $L - 2F'$ is numerically equivalent to an effective divisor. Hence $g(L) \geq g(C) + (K_{X/C} + (n-1)L)L^{n-2}F'$ since $(L_{F'})^{n-1} \geq 1$.

So we have

$$\begin{aligned} g(L) &\geq g(C) + (K_{F'} + (n-1)L_{F'})(L_{F'})^{n-2} \\ &= g(C) + (K_{F'} + (n-2)L_{F'})(L_{F'})^{n-2} + (L_{F'})^{n-1} \\ &= g(C) + 2g(L_{F'}) - 2 + (L_{F'})^{n-1} \\ &= g(C) + g(L_{F'}) + g(L_{F'}) - 2 + (L_{F'})^{n-1}. \end{aligned}$$

If $g(L_{F'}) \geq 1$, then $g(L) \geq g(C) + g(L_{F'})$. So we assume that $g(L_{F'}) = 0$. By Theorem 1.3, $g(L) \geq g(C) = g(C) + g(L_{F'})$.

Case (b): $g(C) = 0$. Let D be an irreducible reduced divisor such that the strict transform of D by θ is a general fiber F'. Then $L - D$ is linearly equivalent to an effective divisor. We may assume that $K_X + (n-1)L$ is nef by Theorem 1.10.

So we have

$$\begin{aligned} g(L) = g(\theta^* L) &= 1 + \frac{1}{2}(K_{X'} + (n-1)\theta^* L)(\theta^* L)^{n-1} \\ &= 1 + \frac{1}{2}\theta^*(K_X + (n-1)L)(\theta^* L)^{n-1} \\ &\geq 1 + \frac{1}{2}\theta^*(K_X + (n-1)L)(\theta^* L)^{n-2}F' \\ &= 1 + \frac{1}{2}(\theta^*(K_X + D) + \theta^*(L - D) + (n-2)\theta^* L)(\theta^* L)^{n-2}F'. \end{aligned}$$

Since $\theta^*(L-D)(\theta^*L)^{n-2}F' \geq 0$, we have

$$g(L) \geq 1 + \frac{1}{2}(\theta^*(K_X+D) + (n-2)\theta^*L)(\theta^*L)^{n-2}F'.$$

By the same argument as in the proof of Claim 2.4, we can prove

$$\theta^*(K_X+D)(\theta^*L)^{n-2}F' \geq (K_{X'}+F')(\theta^*L)^{n-2}F'.$$

Hence

$$\begin{aligned} g(L) &\geq 1 + \frac{1}{2}(K_{X'}+F' + (n-2)\theta^*L)(\theta^*L)^{n-2}F' \\ &= g(\theta^*L_{F'}). \end{aligned}$$

□

Before we study polarized 3-folds (X, L) such that $g(L) = q(X)$, we prove the following Theorem.

Theorem 2.8. *Let (X, L) be a polarized 3-fold with $h^0(L) \geq 2$. We use Notation 2.6. Then*

1. *$g(L) \geq aq(X)$ if $g(C) = 0$,*
2. *$g(L) \geq q(X) + (a-1)q(F')$ if $g(C) \geq 1$,*

unless $\Delta(L) = 0$ or (X, L) is a scroll over a curve.

Proof. If $K_X + 2L$ is not nef, then $\Delta(L) = 0$ or (X, L) is a scroll over a curve by Theorem 1.10. So we may assume that $K_X + 2L$ is nef. Let $Z = \sum_{i=1}^{m} b_i Z_i$, and Z_i' be the strict transform of Z_i by θ. Let $\theta' : X'' \to X'$ be a birational morphism such that Z_i'' is a smooth surface, where Z_i'' is the strict transform of Z_i' by θ'. We can take a general element $B \in |L|$ such that $B = G_1 + \cdots + G_a + Z$, where each G_i is the image of a general fiber of f' by θ. Let $h = f' \circ \theta'$ and $\pi = \theta \circ \theta'$. Then the strict transform of G_i by $\theta \circ \theta'$ is a general fiber of h. Let $\pi = \theta_1 \circ \cdots \circ \theta_u$, where $\theta_i : X_i \to X_{i-1}$ is blowing up at smooth center B_{i-1} on X_{i-1}. Let F_i'' (resp. Z_i'') be the strict transform of G_i (resp. Z_i) by π. Then

$$\begin{aligned} g(L) = g(\pi^*L) &= 1 + \frac{1}{2}(K_{X''} + 2\pi^*L)(\pi^*L)^2 \\ &= 1 + \frac{1}{2}\pi^*(K_X + 2L)(\pi^*L)(\pi^*B) \\ &\geq 1 + \frac{1}{2}\pi^*(K_X+2L)(\pi^*L)(\pi^*(B_{\mathrm{red}})). \end{aligned}$$

Let $B_{\mathrm{nr}} = B - B_{\mathrm{red}}$.

Claim 2.9. $B_{\mathrm{nr}} B_{\mathrm{red}} L \geq 0$.

Proof. If $B_{\mathrm{nr}} = 0$, then this is obvious. So we may assume $B_{\mathrm{nr}} \neq 0$. Since L is ample, $\mathrm{Bs}\,|nL| = \phi$ for large n. We can take a general element $S \in |nL|$ such that S is smooth and $B_{\mathrm{nr}}|_S$ and $B_{\mathrm{red}}|_S$ are nonzero effective divisors. Since L is ample, so is L_S. On the other hand, $L_S = B_{\mathrm{nr}}|_S + B_{\mathrm{red}}|_S$. By Lemma 2 in [Ra], $B_{\mathrm{nr}}|_S B_{\mathrm{red}}|_S > 0$. Therefore $B_{\mathrm{nr}} B_{\mathrm{red}} L > 0$. This completes the proof of this Claim. □

So we obtain

$$\begin{aligned}
g(L) &\geq 1+\frac{1}{2}\pi^*(K_X+2L)(\pi^*L)(\pi^*(B_{\mathrm{red}}))\\
&= 1+\frac{1}{2}(\pi^*(K_X)+\pi^*(B_{\mathrm{red}})+\pi^*(B_{\mathrm{nr}})+\pi^*L)(\pi^*L)(\pi^*(B_{\mathrm{red}}))\\
&\geq 1+\frac{1}{2}(\pi^*(K_X+B_{\mathrm{red}})+\pi^*L)(\pi^*L)(\pi^*(B_{\mathrm{red}}))\\
&= 1+\frac{1}{2}\sum_{i=1}^{a}(\pi^*(K_X+B_{\mathrm{red}})+\pi^*L)(\pi^*L)F_i''\\
&\quad +\frac{1}{2}\sum_{i=1}^{m}(\pi^*(K_X+B_{\mathrm{red}})+\pi^*L)(\pi^*L)Z_i''\\
&= 1+\frac{1}{2}\sum_{i=1}^{a}(\pi^*(K_X+G_i)+\pi^*L)(\pi^*L)F_i''\\
&\quad +\frac{1}{2}\sum_{i=1}^{m}(\pi^*(K_X+Z_i)+\pi^*L)(\pi^*L)Z_i''\\
&\quad +\frac{1}{2}\sum_{i=1}^{a}\pi^*(B_{\mathrm{red}}-G_i)(\pi^*L)F_i''+\frac{1}{2}\sum_{i=1}^{m}\pi^*(B_{\mathrm{red}}-Z_i)(\pi^*L)Z_i''.
\end{aligned}$$

Since L is ample and B is connected,

$$\begin{aligned}
&\frac{1}{2}\Big(\sum_{i=1}^{a}\pi^*(B_{\mathrm{red}}-G_i)(\pi^*L)F_i''+\sum_{i=1}^{m}\pi^*(B_{\mathrm{red}}-Z_i)(\pi^*L)Z_i''\Big)\\
&= \frac{1}{2}\Big((B_{\mathrm{red}})^2-\Big(\sum_{i=1}^{a}G_i^2+\sum_{i=1}^{m}Z_i^2\Big)\Big)L\\
&\geq a+m-1.
\end{aligned}$$

Hence

$$\begin{aligned}
g(L) &\geq 1+\frac{1}{2}\sum_{i=1}^{a}(\pi^*(K_X+G_i)+\pi^*L)(\pi^*L)F_i''\\
&\quad +\frac{1}{2}\sum_{i=1}^{m}(\pi^*(K_X+Z_i)+\pi^*L)(\pi^*L)Z_i''\\
&\quad +(a+m-1)\\
&= \sum_{i=1}^{a}\Big(1+\frac{1}{2}(\pi^*(K_X+G_i)+\pi^*L)(\pi^*L)F_i''\Big)\\
&\quad +\sum_{i=1}^{m}\Big(1+\frac{1}{2}(\pi^*(K_X+Z_i)+\pi^*L)(\pi^*L)Z_i''\Big).
\end{aligned}$$

By the same argument as in the proof of Claim 2.4, we can prove that

$$(\pi^*(K_X+G_i)+\pi^*L)(\pi^*L)F_i''\geq(K_{X''}+F_i''+\pi^*L)(\pi^*L)F_i''$$

and

$$(\pi^*(K_X+Z_i)+\pi^*L)(\pi^*L)Z_i''\geq(K_{X''}+Z_i''+\pi^*L)(\pi^*L)Z_i''.$$

So we obtain that

$$\begin{aligned} g(L) &\geq \sum_{i=1}^{a}(1+\frac{1}{2}(K_{X''}+F''_i+\pi^*L)(\pi^*L)F''_i) \\ &\quad + \sum_{i=1}^{m}(1+\frac{1}{2}(K_{X''}+Z''_i+\pi^*L)(\pi^*L)Z''_i) \\ &= \sum_{i=1}^{a} g(\pi^*L_{F''_i}) + \sum_{i=1}^{m} g(\pi^*L_{Z''_i}). \end{aligned}$$

We remark that $g(\pi^*L_{Z''_i}) \geq 0$ for any i since $\dim Z''_i = 2$.

Case (I). $g(C)=0$. Since $h^0(\pi^*L_{F''_i}) \geq 1$ and $\dim F''_i = 2$, we have $g(\pi^*L_{F''_i}) \geq q(F''_i)$ for any i. Because of $q(F''_i) \geq q(X'') = q(X') = q(X)$ for any i, we obtain that $g(L) \geq aq(X)$.

Case (II). $g(C) \geq 1$.

Then $\theta = \mathrm{id}$. Since L is ample and G_i is a fiber of f', there exists a Z_i such that $Z_i \to C$ is surjective. We consider the fiber space $Z''_i \to C$. By Theorem 2.1 and Theorem 5.5 in [Fk1], $g(\pi^*L_{Z''_i}) \geq g(C)$ for some i. On the other hand, $g(\pi^*L_{F''_i}) \geq q(F''_i)$ because $h^0(\pi^*L_{F''_i}) \geq 1$ and $\dim F''_i = 2$. Hence $g(L) \geq g(C)+aq(F''_i)$. Since $g(C)+q(F''_i) \geq q(X'') = q(X') = q(X)$ and $q(F''_i) = q(F')$ for any i, we obtain that $g(L) \geq q(X)+(a-1)q(F')$. (We remark that $a \geq 2$ in this case.)

This completes the proof of this Theorem. □

By Theorem 1.6 and Theorem 1.13, we can prove the following Theorem by the same method as that used in the proof of Theorem 2.8 (cf. Theorem 2.7).

Theorem 2.10. *Let (X,L) be a polarized 4-fold with $h^0(L) \geq 2$. We use Notation 2.6. Then*

$$g(L) \geq g(C) + ag(\theta^*L_{F'})$$

unless $\Delta(L)=0$ or (X,L) is a scroll over a curve.

In particular, if $a=1$ and $h^0(L) \geq 3$, then $g(L) \geq q(X)$ by Theorem 2.1.

Theorem 2.11. *Let (X,L) be a polarized 3-fold with $h^0(L) \geq 2$. We use Notation 2.6.*

If $a=1$, then $g(L) \geq q(X) + \frac{1}{2}GZL$ unless $\Delta(L)=0$ or (X,L) is a scroll over a curve, where G is a general element of Λ_M.

In particular, $g(L) \geq q(X)+1$ if $Z \neq 0$.

Proof. We remark that the strict transform of G by θ is F'. So we have

$$\begin{aligned} g(L) &= 1+\frac{1}{2}(K_{X'}+2\theta^*L)(\theta^*L)^2 \\ &= 1+\frac{1}{2}\theta^*(K_X+2L)(\theta^*L)^2 \\ &\geq 1+\frac{1}{2}\theta^*(K_X+2L)(\theta^*L)F' \\ &= 1+\frac{1}{2}(\theta^*(K_X+G)+\theta^*(L-G)+\theta^*L)(\theta^*L)F'. \end{aligned}$$

By using the same argument as in the proof of Claim 2.4, we can prove $\theta^*(K_X+G)(\theta^*L)F' \geq (K_{X'}+F')(\theta^*L)F'$. On the other hand, $\theta^*(L-G)(\theta^*L)F' = ZGL$. Hence

$$\begin{aligned} g(L) &\geq 1 + \frac{1}{2}(K_{X'} + F' + \theta^*L)(\theta^*L)F' + \frac{1}{2}ZGL \\ &= g(\theta^*L_{F'}) + \frac{1}{2}ZGL. \end{aligned}$$

Since $h^0(\theta^*L_{F'}) \geq 1$ and $\dim F' = 2$, we obtain that $g(\theta^*L_{F'}) \geq q(F')$.

Therefore

$$\begin{aligned} g(L) &\geq q(F') + \frac{1}{2}ZGL \\ &\geq q(X) + \frac{1}{2}ZGL. \end{aligned}$$

If $Z \neq 0$, then $Z \cap G \neq \phi$ since $G + Z$ is connected. Since L is ample and G is a general element of Λ_M, we have $ZGL > 0$. Because $g(L)$ is integer, $g(L) \geq q(X) + 1$. □

Theorem 2.12. *Let (X, L) be a polarized 3-fold with $h^0(L) \geq 3$. If $g(L) = q(X)$, then (X, L) is one of the following types:*

1. $\Delta(L) = 0$,
2. (X, L) *is a scroll over a curve.*

Proof. We use Notation 2.6.

(Step 1). First we assume that (X, L) is not obtained by a finite number of simple blowing up of another polarized 3-fold.

If $K_X + 2L$ is not nef, then (X, L) is the type (1) or (2). So we may assume that $K_X + 2L$ is nef.

(1-1) The case in which $g(C) \geq 1$.

We remark that $\theta = \mathrm{id}$ in this case. By Theorem 2.8, $q(X) = g(L) \geq q(X) + (a-1)q(F')$. Since $a \geq 2$, we obtain that $q(F') = 0$. So we have $g(L) = q(X) = g(C)$. But then (f', X, C, L) is a scroll by Theorem 1.3. This is a contradiction by assumption.

(1-2) The case in which $g(C) = 0$.

If $a \geq 2$, then $q(X) = g(L) \geq 2q(X)$ by Theorem 2.8. Hence $q(X) = 0$. Therefore $g(L) = q(X) = 0$. By Theorem 1.16, $\Delta(L) = 0$ in this case. This is a contradiction by hypothesis.

So we consider the case $a = 1$. By Theorem 2.11, $Z = 0$, that is, $|L|$ has no fixed component. By the proof of Theorem 2.8, we have $g(\theta^*L_{F'}) = q(F')$. Since $h^0(\theta^*L_{F'}) \geq 2$, we have $\kappa(F') = -\infty$. Since $g(\theta^*L_{F'}) = q(F')$, we can prove the following Claim.

Claim 2.13. $\kappa(K_{F'} + \theta^*L_{F'}) = -\infty$.

Proof. Assume that $\kappa(K_{F'} + \theta^*L_{F'}) \geq 0$. Then $g(\theta^*L_{F'}) \geq 1$.

Since $g(\theta^*L_{F'}) = q(F')$, a $(\theta^*L_{F'})$-minimalization of $(F', \theta^*L_{F'})$ (see Definition 1.17) is a scroll over a curve B by Theorem 3.1 in [Fk1]. Hence there is a surjective morphism $\pi : F' \to B$ such that a general fiber F_π of π is $\mathbb{P}^1$. Hence

$$(K_{F'} + \theta^*L_{F'})F_\pi = -1.$$

But this is a contradiction because F_π is nef. This completes the proof of this claim. □

On the other hand,

$$\begin{aligned} K_{F'} + \theta^* L_{F'} &= (K_{X'} + F' + \theta^* L)_{F'} \\ &= (\theta^*(K_X + L) + E_\theta + F')_{F'}, \end{aligned}$$

where E_θ is a θ-exceptional effective divisor.

Since F' is a general fiber of f', we have $h^0((E_\theta + F')|_{F'}) \geq 1$.

If $K_X + L$ is nef, $m(K_X + L)$ has no base point by the Base Point Free Theorem (see [KMM]) for large m. Hence $h^0(m(\theta^*(K_X + L)_{F'})) \geq 1$. Therefore

$$h^0(m((\theta^*(K_X + L) + E_\theta + F')|_{F'})) > 0.$$

But this is a contradiction by Claim 2.13.

Therefore $K_X + L$ is not nef. Hence we use Theorem 1.12 and its notation.

By assumption of (Step 1), the case (a) of Theorem 1.12 cannot occur.

If (X, L) is the case (b0), then $g(L) = q(X) = 0$. Therefore $\Delta(L) = 0$ by Theorem 1.16 and this is a contradiction by hypothesis.

If (X, L) is the case (b1), then $g(L) = q(X) = g(W)$. But by Theorem 1.3, (X, L) is a scroll over W. This is a contradiction by hypothesis.

So we consider the case in which (X, L) is the type (b2).

Let $\varphi : X \to Y$ be its $\mathbb{P}^1$-bundle, where Y is a smooth surface.

Claim 2.14. $\kappa(Y) = -\infty$.

Proof. We remark that $Z = 0$. We take a general element $G \in |L|$. Then G is irreducible and reduced, and the strict transform of G by θ is F'. Since L is ample, $\varphi|_G : G \to Y$ is surjective. Hence we obtain $\kappa(Y) = -\infty$ because $\kappa(F') = -\infty$. This completes the proof of this Claim. □

If $q(Y) = 0$, then $g(L) = q(X) = q(Y) = 0$. Hence $\Delta(L) = 0$ by Theorem 1.16. But this is a contradiction by assumption.

If $q(Y) \geq 1$, we take the Albanese map of Y $\alpha : Y \to B$, where B is a smooth curve. Then $g(L) = q(X) = q(Y) = g(B)$. Hence by Theorem 1.3, $(\alpha \circ \pi, X, B, L)$ is a scroll. But this is a contradiction by assumption.

(Step 2). Next we assume that (X, L) is obtained by a finite number of simple blowing up of another polarized 3-fold (Y, A) which is not obtained by a finite number of simple blowing up of another polarized 3-fold.

Then by (Step 1), (Y, A) is one of the following types because $g(L) = g(A)$ and $q(X) = q(Y)$:

(A) $\Delta(A) = 0$,

(B) (Y, A) is a scroll over a curve.

Let $\pi : X \to Y$ be its birational morphism.

If (Y, A) is the type (A), then $g(A) = 0$. Since $g(L) = g(A)$, we have $g(L) = 0$ and $\Delta(L) = 0$ by Theorem 1.16. So we assume that (Y, A) is the type (B).

Let $h : Y \to B$ be its $\mathbb{P}^2$-bundle. Then $g(A) = g(B)$. We consider the fiber space $(h \circ \pi, X, B)$. Since $g(L) = g(A)$, we have $g(L) = g(B)$. But then $(h \circ \pi, X, B, L)$ is a scroll by Theorem 1.3. But this is a contradiction because $\pi \neq \mathrm{id}$. This completes the proof of Theorem 2.12. □

By considering Theorem 2.12, we propose the following conjecture.

Conjecture 2.15. *Let (X, L) be a polarized manifold with $\dim X = n \geq 4$ and $h^0(L) \geq n$. If $g(L) = q(X)$, then (X, L) is one of the following types:*

1. $\Delta(L) = 0$,
2. *(X, L) is a scroll over a smooth curve.*

References

[B1] L. Bădescu, *On ample divisors*, Nagoya Math. J. **86** (1982), 155–171. MR **83j:**14008

[B2] ———, *On ample divisors II*, Proceedings of the "Week of Algebraic Geometry" (Bucharest) 1980 Teubner Texte Math **40** (1981), 12–32. MR **84k:**14004

[B3] ———, *The projective plane blown-up at a point as an ample divisor*, Atti. Accad. Ligure **38** (1981), 3–7. MR **85e:**14055

[B-S] M. C. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties*, de Gruytev Expositions in Math. **16**. MR **96f:**14004

[Fj0] T. Fujita, *Classification Theories of Polarized Varieties*, London Math. Soc. Lecture Note Series **155** (1990). MR **93e:**14009

[Fj1] ———, *On polarized manifolds whose adjoint bundles are not semipositive*, Advanced Studies in Pure Math **10** (1985), 167–178. MR **89d:**14006

[Fj2] ———, *Remarks on quasi-polarized varieties*, Nagoya Math. J **115** (1989), 105–123. MR **90i:**14045

[Fk1] Y. Fukuma, *A lower bound for the sectional genus of quasi-polarized surfaces*, Geometriae Dedicata **64** (1997), 229–251. MR **98c:**14007

[Fk2] ———, *A lower bound for sectional genus of quasi-polarized manifolds*, J. Math. Soc. Japan **49** (1997), 339–362. CMP 98:07

[Fk3] ———, *A lower bound for sectional genus of quasi-polarized manifolds II*, preprint.

[Fk4] ———, *On sectional genus of quasi-polarized manifolds with non-negative Kodaira dimension*, Math. Nachr. **180** (1996), 75–84. MR **97f:**14005

[I] P. Ionescu, *Generalized adjunction and applications*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 457–472. MR **87e:**14031

[KMM] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model problem*, Advanced Studies in Pure Math **10** (1985), 283–360. MR **89e:**14015

[Ma] H. Maeda, *On polarized surfaces of sectional genus three*, Sci. Papers College Arts Sci. Univ. Tokyo **37** (1987), 103–112. MR **89h:**14027

[Ra] C. P. Ramanujam, *Remarks on the Kodaira vanishing theorem*, J. Indian Math. Soc. **36** (1972), 41–51. MR **48:**8502

[S] A. J. Sommese, *On the adjunction theoretic structure of projective varieties*, Lecture Notes in Math **1194**, 175–213. MR **87m:**14049

[S-V] A. J. Sommese and Van de Ven, *On the adjunction mapping*, Math. Ann. **278** (1987), 593–603. MR **88j:**14011

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